113 Class Problems: Finitely Generated Groups

1. Consider the group $(\mathbb{C} \setminus \{0\}, \times)$.
   (a) Prove that for any $m \in \mathbb{N}$, there exists a cyclic subgroup of $\mathbb{C} \setminus \{0\}$ of size m.
   (b) Prove that this subgroup is the unique such subgroup.
   (c) List all possible single set generators for this subgroup.

   Solution:

a)

Assume $\exists\, x \in \mathbb{C} \setminus \{0\}$ such that $|gp(\{x\})| = m$

$\Leftrightarrow \quad ord(x) = m \quad \Leftrightarrow \quad x^m = 1$ and $x^d \neq 1 \quad \forall \quad 0 < d < m$

$re^{i\theta} = 1 \quad \Leftrightarrow \quad r = 1$ and $\theta = 2\pi k$ where $k \in \mathbb{Z}$

If $x = re^{i\theta}$ and $x^m = 1 \quad \Rightarrow \quad r^m e^{im\theta} = 1 \quad \Rightarrow r = 1$ and $m\theta = 2\pi k$ for some $k \in \mathbb{Z}$

$\Rightarrow \quad x = e^{i\frac{2\pi k}{m}}$ for $k \in \mathbb{Z}$

Observe $\left(e^{i\frac{2\pi}{m}}\right)^d = e^{\frac{i 2\pi d}{m}} \neq 1 \quad \forall \quad 0 < d < m$, hence

$ord\left(e^{i\frac{2\pi}{m}}\right) = m \quad \Rightarrow \quad \left|gp\left(\{e^{\frac{2\pi i}{m}}\}\right)\right| = m$

$\left(gp\left(\{e^{\frac{2\pi i}{m}}\}\right) = \left\{ e^{\frac{2\pi i k}{m}} \mid k \in \mathbb{Z} \right\}\right)$ all solutions of $x^m = 1$ in $\mathbb{C}$

$\{1, e^{\frac{2\pi i}{m}}, e^{\frac{2\pi i}{m}\cdot 2}, \dots, e^{\frac{2\pi i}{m}(m-1)}\}$

vertices of regular m-gon centered at origin

b) If $H \subset \mathbb{C} \setminus \{0\}$ is a cyclic subgroup, $|H| = m$

$\Rightarrow \quad \exists\, y \in H$ such that $y^m = 1$ and $gp(\{y\}) = H$

$\Rightarrow \quad y \in gp\left(\{e^{\frac{2\pi i}{m}}\}\right) \quad \Rightarrow \quad H \subset gp\left(\{e^{\frac{2\pi i}{m}}\}\right) \quad \Rightarrow \quad H = gp\left(\{e^{\frac{2\pi i}{m}}\}\right)$

c) Let $e^{\frac{2\pi i k}{m}} \in gp\left(e^{\frac{2\pi i}{m}}\right)$ be a generator

$\Leftrightarrow \quad \left(e^{\frac{2\pi i k}{m}}\right)^d \neq 1 \quad \forall \quad 0 < d < m$

$\Leftrightarrow \quad \frac{kd}{m} \notin \mathbb{Z} \quad \forall \quad 0 < d < m$

$\Leftrightarrow \quad HCF(k, m) = 1$, ie $e^{\frac{2\pi i k}{m}}$ is a generator $\Leftrightarrow$ k, m coprime

2. Prove that $(\mathbb{Q}, +)$ is not finitely generated.

Solution:

Let $\{\frac{a_1}{b_1}, \ldots \frac{a_n}{b_n}\} \subset \mathbb{Q}$

$\Rightarrow gp(\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\}) = \{\lambda_1 \frac{a_1}{b_1} + \ldots + \lambda_n \frac{a_n}{b_n} \mid \lambda_i \in \mathbb{Z}\}$

$\lambda_1 \frac{a_1}{b_1} + \ldots + \lambda_n \frac{a_n}{b_n} = \frac{c}{b_1 \ldots b_n}$ for some $c \in \mathbb{Z}$

$\Rightarrow \frac{1}{b_1 \ldots b_n + 1} \notin gp(\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\})$

$\Rightarrow gp(\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\}) \neq \mathbb{Q} \Rightarrow (\mathbb{Q}, +)$ not finitely generated

3. Let $(G, *)$ be a group and $x \in G$ such that $G = gp(\{x\})$. Let H be a second group and $\phi, \psi : G \to H$ be two homomorphisms. Recall that $\phi = \psi$ means that $\phi(g) = \psi(g)$ for all $g \in G$. Prove the following:

$$\phi = \psi \iff \phi(x) = \psi(x).$$

Solution:

($\Rightarrow$) trivial

($\Leftarrow$)

Let $y \in G \Rightarrow y = x^a$ for some $a \in \mathbb{Z}$

ϕ a homomorphism

ψ a homomorphism

$\Rightarrow \phi(y) = \phi(x^a) = (\phi(x))^a = (\psi(x))^a = \psi(x^a) = \psi(y)$